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2001 J. Phys. A: Math. Gen. 34 8543

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# Integrable Kondo impurity in one-dimensional $q$ -deformed $t$ - $J$ models

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Received 5 September 2000, in final form 16 August 2001

Published 5 October 2001

Online at [stacks.iop.org/JPhysA/34/8543](http://stacks.iop.org/JPhysA/34/8543)

## Abstract

Integrable Kondo impurities in two cases of one-dimensional  $q$ -deformed  $t$ - $J$  models are studied by means of the boundary  $\mathbb{Z}_2$ -graded quantum inverse scattering method. The boundary  $K$  matrices depending on the local magnetic moments of the impurities are presented as nontrivial realizations of the reflection equation algebras in an impurity Hilbert space. Furthermore, these models are solved by using the algebraic Bethe ansatz method and the Bethe ansatz equations are obtained.

PACS numbers: 71.10.Fd, 71.27.+a, 75.10.Jn

## 1. Introduction

The Kondo problem describing the effect due to the exchange interaction between magnetic impurities and the conduction electrons plays a very important role in condensed matter physics [1]. Wilson [2] developed a very powerful numerical renormalization group approach, and the model was also solved by the coordinate Bethe ansatz method [3, 4] which gives the specific heat and magnetization. More recently, a conformal field theory approach was developed by Affleck and Ludwig [5] based on work by Nozières [6]. In the conventional Kondo problem, the interaction between conduction electrons is discarded, due to the fact that the interacting electron system can be described by a Fermi liquid. Recently there has been substantial research devoted to the investigation of the theory of impurities coupled to Luttinger liquids. Such a problem was first considered by Lee and Toner [7]. In order to get a full picture about the critical behaviour of Kondo impurities coupled to Luttinger liquids, some simple integrable models which allow exact solutions are desirable.

Several integrable impurity problems in Luttinger liquids describing impurities embedded in systems of correlated electrons have so far appeared in the literature. Among them are versions of the supersymmetric  $t$ - $J$  model with impurities [8–13]. Such an idea (to incorporate an impurity into a closed chain) dates back to Andrei and Johansson [14] and Lee and

Schlottmann [15] (see also [16]). However, the models thus constructed suffer from the lack of backward scattering and result in a very complicated Hamiltonian which is difficult to justify on physical grounds. Therefore, as observed by Kane and Fisher [17], it is advantageous to adopt open boundary conditions with the impurities situated at the ends of the chain when studying Kondo impurities coupled to integrable strongly correlated electron systems [18].

In our earlier work [19, 20] we were able to derive in an algebraic fashion integrable boundary Kondo impurities for the isotropic supersymmetric  $t$ - $J$  model. In this paper, integrable Kondo impurities with spin- $\frac{1}{2}$  coupled to the one-dimensional  $q$ -deformed  $t$ - $J$  open chain are constructed following our earlier formalism. Our new input is to search for integrable boundary  $K$  matrices depending on the local magnetic moments of impurities, which arise as a nontrivial realization of the  $Z_2$ -graded reflection equation (RE) algebras in a finite-dimensional quantum space, which may be interpreted as an impurity Hilbert space. It should be emphasized that our new non- $c$ -number boundary  $K$  matrices are highly nontrivial, in the sense that they cannot be factorized into the product of a  $c$ -number boundary  $K$  matrix and the corresponding local monodromy matrices. The models we present are solved by means of the algebraic Bethe ansatz method and the Bethe ansatz equations are derived.

Recently, the work of Frahm and Slavnov [23] has provided a representation theoretic explanation for the existence of these non-regular solutions of the reflection equations. These solutions arise as a projection of a regular solution (subject to some consistency requirements) onto a subspace of the associated impurity Hilbert space. The projection method has the effect of reducing the local symmetry of the projected boundary operator to some subalgebra of the symmetry algebra of the original regular solution. This result is entirely consistent with our findings here for the existence of integrable boundary Kondo impurities in  $q$ -deformed  $t$ - $J$  models.

Before going further, let us make some comments about the relationship between our construction and others that have appeared in the literature [8, 9, 24, 25]. In these works, integrable Kondo-like magnetic impurities were studied in the closed  $t$ - $J$  and Hubbard chains. Unfortunately, the arguments in these papers do not appear to be mathematically sound. Here, we will address in particular the algebraic approach adopted in [24, 25] for the case of the  $t$ - $J$  model. In these cases, the author appealed to the quantum inverse scattering method (QISM), claiming that the Hamiltonians are derivable from the transfer matrix, without presenting the impurity monodromy matrix. However, in our opinion, it is reasonable to question the existence of such an impurity monodromy matrix in view of the form of the Bethe ansatz solutions obtained. A standard calculation shows that the presence of the impurity changes the pseudovacuum eigenvalues, but does not affect the fundamental commutation relations of the underlying Yangian algebra. In [24], and in particular the appendix to [25], it is claimed that the operators  $\hat{A}_{12}$ ,  $\hat{A}_{21}$  both vanish on the pseudovacuum in the solution of the  $t$ - $J$  model with impurities in the FFB grading. This implies that the pseudovacuum provides a one-dimensional representation of the  $su(2)$  sub-Yangian (let us call it  $\mathcal{A}$ ) generated by  $\hat{A}_{11}$ ,  $\hat{A}_{22}$ ,  $\hat{A}_{12}$ ,  $\hat{A}_{21}$ , which in turn implies that the operators  $\hat{A}_{11}$ ,  $\hat{A}_{22}$  must take the same eigenvalue on the pseudovacuum. It is clear from the transfer matrix eigenvalues presented in [24, 25] that this does not occur in those works. Furthermore, it means that the effect of the impurity only changes the first level Bethe ansatz equations, leaving the second level nested equations unchanged. Inspection of the Bethe ansatz equations given in [24, 25] again shows an inconsistency since in those papers it is clear that the impurity affects the Bethe ansatz equations at both levels.

On the other hand, if the impurity in the monodromy matrix is coming from a higher-dimensional atypical representation of  $gl(2|1)$  with a shift in the spectral parameter, as seems to be the case in [24, 25], then the operator  $\hat{A}_{21}$  *does not* vanish on the pseudovacuum chosen

there. In this instance, it is necessary to use, for the Bethe ansatz procedure, a subspace of pseudovacuum states which are stable under the action of the  $su(2)$  sub-Yangian  $\mathcal{A}$ , as was adopted in [10, 11, 13] for other impurity  $t$ - $J$  models. A  $(2l + 1)$ -dimensional atypical representation of  $gl(2|1)$  decomposes into an  $(l + 1)$ -dimensional (spin  $S = l/2$ ) and an  $l$ -dimensional (spin  $S = (l - 1)/2$ ) representation with respect to the  $su(2)$  subalgebra. Only in the case of the fundamental representation ( $l = 1$ ) is there a singlet state with respect to  $su(2)$  on which  $\hat{A}_{21}$  will vanish. Performing the Bethe ansatz with these considerations does not reproduce the transfer matrix eigenvalues and Bethe ansatz equations given in [24, 25].

As shown in our work [19–22], for the  $t$ - $J$  model  $R$  matrix and Shastry's  $R$  matrix for the Hubbard model, there are no such local impurity monodromy matrices to guarantee that their electron-impurity scattering matrices can be inferred from the corresponding Hamiltonians [20]. Specifically, although for the  $t$ - $J$  model there is a singular local monodromy matrix, no such local monodromy matrix exists for the Hubbard model. Also, even for the  $t$ - $J$  model, the singularity of such a local monodromy matrix does not allow us to use it to construct a closed  $t$ - $J$  chain interacting with magnetic impurities in a closed chain. This conclusion was confirmed in [23].

The above argument also clearly indicates that our construction is completely different from that of [24, 25], because there it was claimed that the impurity position in an open chain is immaterial, that is, the Kondo impurity may be put either in the bulk or at boundaries [24]. Moreover, it was claimed the impurity is a forward scatterer [26], which is in contrast to our results. As shown in [19, 20], for the  $t$ - $J$  model, such a singular local monodromy matrix can be used to well-define a non- $c$ -number boundary  $K$  matrix which leads us to integrable Kondo-like impurities in the corresponding open chains. This shows the important result that this Kondo impurity is completely backward scattering.

The layout of this paper is as follows. We begin by reviewing the  $\mathcal{Z}_2$ -graded boundary QISM as formulated in [27, 28]. We then introduce two integrable cases of the one-dimensional  $q$ -deformed  $t$ - $J$  model with Kondo impurities on the boundaries. Integrability of the models is established by relating the Hamiltonians to one-parameter families of commuting transfer matrices. This is achieved through solving the reflection equations for non- $c$ -number solutions. Finally we solve the models by means of the algebraic Bethe ansatz method and derive the Bethe ansatz equations.

## 2. Graded reflection equation algebra and transfer matrix

In this section, we give a brief review of the  $\mathcal{Z}_2$ -graded boundary QISM. To begin, let  $V$  be a finite-dimensional  $\mathcal{Z}_2$ -graded linear superspace and let the operator  $R(u)$  satisfy the graded quantum Yang–Baxter equation

$$R_{12}(u_1 - u_2)R_{13}(u_1)R_{23}(u_2) = R_{23}(u_2)R_{13}(u_1)R_{12}(u_1 - u_2).$$

Here  $R_{jk}(u)$  denotes the matrix on  $V \otimes V \otimes V$  acting on the  $j$ th and  $k$ th superspaces and as an identity on the remaining superspace. The standard notation is used with  $R_{12}(u) = R(u) \otimes I$ ,  $R_{23}(u) = I \otimes R(u)$ , etc, where  $R(u) = \sum_i a_i \otimes b_i \in \text{End}(V \otimes V)$ . The variables  $u_1$  and  $u_2$  are spectral parameters. The tensor product should be understood in the graded sense, that is, the multiplication rule for any homogeneous elements  $x, y, x', y' \in \text{End} V$  is given by

$$(x \otimes y)(x' \otimes y') = (-1)^{[y][x']} (xx' \otimes yy') \quad (2.1)$$

where  $[x]$  stands for the  $\mathcal{Z}_2$ -grading of the element  $x$ . Let  $P$  be the  $\mathcal{Z}_2$ -graded permutation operator in  $V \otimes V$ . Then  $P(x \otimes y) = (-1)^{[x][y]} y \otimes x$ ,  $\forall x, y \in V$  and  $R_{21}(u) = P_{12}R_{12}(u)P_{12}$ .

We form the monodromy matrix  $T(u)$  for a  $L$ -site lattice chain by

$$T(u) = R_{0L}(u) \cdots R_{01}(u)$$

$$(T(u)^{ab})_{\alpha_1, \beta_1, \dots, \alpha_L, \beta_L} = R_{0L}(u)_{\alpha_L \beta_L}^{a c_L} R_{0L-1}(u)_{\alpha_{L-1} \beta_{L-1}}^{c_L c_{L-1}} \cdots R_{01}(u)_{\alpha_1 \beta_1}^{c_2 b} (-1)^{\sum_{j=2}^L ([\alpha_j] + [\beta_j]) \sum_{i=1}^{j-1} [\alpha_i]}$$

where 0 still represents the auxiliary superspace, and the tensor product is still in the graded sense.  $T(u)$  is a quantum operator valued matrix that acts nontrivially in the graded tensor product of all quantum superspaces of the lattice.

Indeed, one may show that  $T(u) \in \text{End}(V \otimes W)$ ,  $R(u) \in \text{End}(V \otimes V)$  generates a representation of the graded quantum Yang–Baxter algebra

$$R_{12}(u_1 - u_2) \overset{1}{T}(u_1) \overset{2}{T}(u_2) = \overset{2}{T}(u_2) \overset{1}{T}(u_1) R_{12}(u_1 - u_2) \quad (2.2)$$

where for notational convenience we have

$$\overset{1}{T}(u) = T_{13}(u) \quad \overset{2}{T}(u) = T_{23}(u)$$

and the subscript 3 now labels the quantum superspace  $W = V^{\otimes L}$ .

In order to describe integrable Kondo impurities in strongly correlated electronic models with open boundary conditions, we need to introduce an appropriate  $\mathbb{Z}_2$ -graded RE algebra. We introduce the associative superalgebras  $\mathcal{T}_-$  and  $\mathcal{T}_+$  defined by the  $R$  matrix and the relations

$$R_{12}(u_1 - u_2) \overset{1}{T}_-(u_1) R_{21}(u_1 + u_2) \overset{2}{T}_-(u_2) = \overset{2}{T}_-(u_2) R_{12}(u_1 + u_2) \overset{1}{T}_-(u_1) R_{21}(u_1 - u_2) \quad (2.3)$$

and

$$\begin{aligned} R_{21}^{st_1 ist_2}(-u_1 + u_2) \overset{1}{T}_+^{st_1}(u_1) \tilde{R}_{12}(-u_1 - u_2) \overset{2}{T}_+^{ist_2}(u_2) \\ = \overset{2}{T}_+^{ist_2}(u_2) \tilde{R}_{21}(-u_1 - u_2) \overset{1}{T}_+^{st_1}(u_1) R_{12}^{st_1 ist_2}(-u_1 + u_2) \end{aligned} \quad (2.4)$$

where we have defined new objects  $\tilde{R}$  and  $\tilde{R}$  through the relations

$$\begin{aligned} \tilde{R}_{12}^{st_2}(-u_1 - u_2) R_{21}^{st_1}(u_1 + u_2) &= 1 \\ \tilde{R}_{21}^{ist_1}(-u_1 - u_2) R_{12}^{ist_2}(u_1 + u_2) &= 1 \end{aligned} \quad (2.5)$$

and  $st_i$  stands for the supertransposition taken in the  $i$ th space, whereas  $ist_i$  is the inverse operation of  $st_i$ . One of the important steps towards formulating a correct formalism for the  $\mathbb{Z}_2$ -graded case is to introduce in equation (2.4) the inverse operation of the supertransposition. In any case, the  $R$  matrices enjoy the unitarity property

$$R_{12}(u_1 - u_2) R_{21}(-u_1 + u_2) = 1. \quad (2.6)$$

One can obtain a class of realizations of the superalgebras  $\mathcal{T}_+$  and  $\mathcal{T}_-$  by choosing  $\mathcal{T}_{\pm}(u)$  to be of the form

$$\mathcal{T}_-(u) = T_-(u) \tilde{T}_-(u) T_-^{-1}(-u) \quad \mathcal{T}_+^{st}(u) = T_+^{st}(u) \tilde{T}_+^{st}(u) (T_+^{-1}(-u))^{st} \quad (2.7)$$

with

$$T_-(u) = R_{0M}(u) \cdots R_{01}(u) \quad T_+(u) = R_{0L}(u) \cdots R_{0, M+1}(u) \quad \tilde{T}_{\pm}(u) = K_{\pm}(u)$$

where  $M$  is any index between 1 and  $L$ , and  $K_{\pm}(u)$ , called boundary  $K$  matrices, are representations of  $\mathcal{T}_{\pm}$ . In the following, without loss of generality, we shall choose  $M = L$  so that  $\mathcal{T}_+(u) \equiv K_+(u)$ .

The  $K$  matrices  $K_{\pm}(u)$  satisfy the same relations as  $\mathcal{T}_{\pm}(u)$  in (2.3) and (2.4), respectively. That is, the  $K$  matrices obey the following graded reflection equations:

$$R_{12}(u_1 - u_2) \overset{1}{K}_-(u_1) R_{21}(u_1 + u_2) \overset{2}{K}_-(u_2) = \overset{2}{K}_-(u_2) R_{12}(u_1 + u_2) \overset{1}{K}_-(u_1) R_{21}(u_1 - u_2) \quad (2.8)$$

and

$$\begin{aligned}
 R_{21}^{st_1 ist_2}(-u_1 + u_2) K_+^{st_1}(u_1) \tilde{R}_{12}(-u_1 - u_2) K_+^{ist_2}(u_2) \\
 = K_+^{ist_2}(u_2) \tilde{R}_{21}(-u_1 - u_2) K_+^{st_1}(u_1) R_{12}^{st_1 ist_2}(-u_1 + u_2).
 \end{aligned}
 \tag{2.9}$$

Now we rewrite the relations (2.5):

$$\begin{aligned}
 \tilde{R}_{21}^{ist_1 st_2}(-u_1 - u_2) &= (((R_{21}(-u_1 - u_2)^{-1})^{st_2^{-1}})^{-1})^{st_2} \\
 \tilde{R}_{12}^{ist_1 st_2}(-u_1 - u_2) &= (((R_{12}(-u_1 - u_2)^{-1})^{st_1})^{-1})^{ist_1}.
 \end{aligned}$$

With the unitarity property (2.6), one can show that the quantity  $\mathcal{T}_+^{st}(u)$  given by (2.7) satisfies equation (2.4) as the following form:

$$\begin{aligned}
 R_{21}^{st_1 ist_2}(-u_1 + u_2) \mathcal{T}_+^{st_1}(u_1) \{ [R_{21}^{st_1}(u_1 + u_2)]^{-1} \}^{ist_2} \mathcal{T}_+^{ist_2}(u_2) \\
 = \mathcal{T}_+^{ist_2}(u_2) \{ [R_{12}^{ist_2}(u_1 + u_2)]^{-1} \}^{st_1} \mathcal{T}_+^{st_1}(u_1) R_{12}^{st_1 ist_2}(-u_1 + u_2).
 \end{aligned}
 \tag{2.10}$$

The  $K$  matrix  $K_+(u)$  satisfies the same relation as  $\mathcal{T}_+(u)$  in (2.10), which fulfils the following graded RE:

$$\begin{aligned}
 R_{21}^{st_1 ist_2}(-u_1 + u_2) K_+^{st_1}(u_1) \{ [R_{21}^{st_1}(u_1 + u_2)]^{-1} \}^{ist_2} K_+^{ist_2}(u_2) \\
 = K_+^{ist_2}(u_2) \{ [R_{12}^{ist_2}(u_1 + u_2)]^{-1} \}^{st_1} K_+^{st_1}(u_1) R_{12}^{st_1 ist_2}(-u_1 + u_2).
 \end{aligned}
 \tag{2.11}$$

Following Sklyanin’s approach in [31], one defines the boundary transfer matrix  $\tau(u)$  as

$$\begin{aligned}
 \tau(u) &= \text{str}_0(\mathcal{T}_+(u)\mathcal{T}_-(u)) \\
 &= \text{str}_0(K_+(u)T(u)K_-(u)T^{-1}(-u)).
 \end{aligned}
 \tag{2.12}$$

Then it can be shown that [28]

$$[\tau(u_1), \tau(u_2)] = 0.$$

### 3. Integrable non- $c$ -number boundary $K$ matrices and Kondo impurities in the one-dimensional $t$ - $J$ models

Let the operators  $c_{j,\sigma}$  and  $c_{j,\sigma}^\dagger$  denote the annihilation and creation operators, respectively, of an electron with spin  $\sigma$  on a lattice site  $j$ , we assume the total number of lattice sites is  $L$  and  $\sigma = \downarrow, \uparrow$  represent spin down and up, respectively. These operators are canonical Fermi operators satisfying anticommutation relations  $\{c_{j,\sigma}^\dagger, c_{j,\tau}\} = \delta_{ij}\delta_{\sigma\tau}$ . We denote by  $n_{j,\sigma} = c_{j,\sigma}^\dagger c_{j,\sigma}$  the number operator for the electron on a site  $j$  with spin  $\sigma$ , and by  $n_j = n_{j,\downarrow} + n_{j,\uparrow}$  the number operator for the electron on a site  $j$ . The Fock vacuum state  $|0\rangle$  satisfies  $c_{j,\sigma}|0\rangle = 0$ .

We consider the following type of Hamiltonians describing two magnetic impurities coupled to a one-dimensional  $q$ -deformed supersymmetric  $t$ - $J$  open chain with  $U_q(gl(2|1))$  symmetry. Make the identifications

$$|1\rangle = c_{j,\downarrow}^\dagger|0\rangle \quad |2\rangle = c_{j,\uparrow}^\dagger|0\rangle \quad |3\rangle = |0\rangle.
 \tag{3.1}$$

Then

$$\begin{aligned}
 H = - \sum_{j=1}^{N-1} \left\{ \sum_{\sigma=\uparrow,\downarrow} (c_{j,\sigma}^\dagger c_{j+1,\sigma} + \text{h.c.}) (1 - n_{j,-\sigma})(1 - n_{j+1,-\sigma}) - S_j^+ S_{j+1}^- - S_j^- S_{j+1}^+ \right. \\
 \left. - q^{-1} n_{j,\uparrow} (1 - n_{j+1,\downarrow}) - q n_{j+1,\uparrow} (1 - n_{j,\downarrow}) - q (n_{j,\downarrow} + n_{j+1,\downarrow}) \right\} \\
 + J_a ((q - q^{-1}) n_{1,\downarrow} - (\sigma_a^- S_1^- + \sigma_a^+ S_1^+) + \sigma_a^z (q n_{1,\uparrow} - q^{-1} n_{1,\downarrow})) + V_a n_1 \\
 + J_b ((q - q^{-1}) n_{N,\downarrow} - (\sigma_b^- S_N^- + \sigma_b^+ S_N^+) + \sigma_b^z (q n_{N,\uparrow} - q^{-1} n_{N,\downarrow})) + V_b n_N.
 \end{aligned} \tag{3.2}$$

Here  $S_j^+, S_j^-$  as usual is the vector spin operator for the conduction electrons at site  $j$ , expressed as  $S_j^+ = c_{j,\uparrow}^\dagger c_{j,\downarrow}$ ,  $S_j^- = c_{j,\downarrow}^\dagger c_{j,\uparrow}$ ;  $\sigma_g^\pm = \sigma_g^x \pm i\sigma_g^y$ ,  $\sigma_g^z$  ( $g = a, b$ ) are the local moments with spin- $\frac{1}{2}$  located at the left and right ends of the system, respectively. The Kondo coupling constants  $J_g, V_g$  ( $g = a, b$ ) at the left and right ends of the chain are expressed in terms of the arbitrary parameters  $c_g$  in the form

$$J_g = \frac{q^{c_g+2}(q - q^{-1})^2}{2(q^{c_g} - q^2)(q^{c_g+2} - 1)} \quad V_g = \frac{q^{c_g+2}(q - q^{-1})(q^2 - 2q^{c_g} + 1)}{2(q^{c_g} - q^2)(q^{c_g+2} - 1)}.$$

It has been shown in [29] that the bulk Hamiltonian acquires an underlying supersymmetry algebra given by  $U_q(gl(2|1))$  in the minimal representation. Throughout we will refer to this case as the supersymmetric  $t$ - $J$  model. Integrability of this model on the open chain with free boundary conditions was established by Foerster and Karowski [29] by showing that the model can be constructed using the QISM. Furthermore, open chain integrability with appropriate boundary conditions was shown in [30].

It is quite interesting to note that, although the introduction of integrable impurities we propose below spoils the supersymmetry, there still remains  $U_q(su(3))$  symmetry in the Hamiltonian (3.3) which maintains conservation of total spin and electron number. We will also establish the quantum integrability of the following Hamiltonian:

$$\begin{aligned}
 H = - \sum_{j=1}^{N-1} \left\{ \sum_{\sigma=\uparrow,\downarrow} (c_{j,\sigma}^\dagger c_{j+1,\sigma} + \text{h.c.}) (1 - n_{j,-\sigma})(1 - n_{j+1,-\sigma}) + S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+ \right. \\
 \left. - q^{-1} n_{j,\uparrow} (1 - n_{j+1,\downarrow}) - q n_{j+1,\uparrow} (1 - n_{j,\downarrow}) - q (n_{j,\downarrow} + n_{j+1,\downarrow}) \right. \\
 \left. + (q + q^{-1})(n_{j,\downarrow} n_{j+1,\downarrow} + n_{j,\uparrow} n_{j+1,\uparrow}) \right\} \\
 + J_a ((q - q^{-1}) n_{1,\downarrow} - (\sigma_a^- S_1^- + \sigma_a^+ S_1^+) + \sigma_a^z (q^{-1} n_{1,\uparrow} - q n_{1,\downarrow})) + V_a n_1 \\
 + J_b ((q - q^{-1}) n_{N,\downarrow} - (\sigma_b^- S_N^- + \sigma_b^+ S_N^+) + \sigma_b^z (q^{-1} n_{N,\uparrow} - q n_{N,\downarrow})) + V_b n_N.
 \end{aligned} \tag{3.3}$$

In this case the dependence of the Kondo coupling constants  $J_g, V_g$  ( $g = a, b$ ) depending on free parameters  $c_g$  takes the form

$$J_g = \frac{q^{c_g+2}(q - q^{-1})^2}{2(q^{c_g} - q^2)(q^{c_g+2} - 1)} \quad V_g = \frac{q^{c_g}(q - q^{-1})(q^2 - 2q^{c_g-2} + 1)}{2(q^{c_g} - q^2)(q^{c_g+2} - 1)}.$$

Let us recall that the local Hamiltonian of the supersymmetric  $q$ -deformed  $t$ - $J$  model is

derived from an  $R$  matrix satisfying the Yang–Baxter equation which has the form [29]

$$R(u) = \begin{pmatrix} q^{u+2} - 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q(q^u - 1) & 0 & -q^u(1 - q^2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q(q^u - 1) & 0 & 0 & 0 & q^u(1 - q^2) & 0 & 0 \\ 0 & -(1 - q^2) & 0 & q(q^u - 1) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{u+2} - 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q(q^u - 1) & 0 & q^u(1 - q^2) & 0 \\ 0 & 0 & 1 - q^2 & 0 & 0 & 0 & q(q^u - 1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 - q^2 & 0 & q(q^u - 1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^u - q^2 \end{pmatrix} \quad (3.4)$$

where  $u$  is the spectral parameter, and we chose to adopt the fermionic, fermionic and bosonic (FFB) grading that means  $[|1\rangle] = [|\bar{2}\rangle] = 1$ ,  $[|\bar{3}\rangle] = 0$  on the indices labelling the basis vectors.

We now solve (2.8) and (2.11) for  $K_-(u)$  and  $K_+(u)$ . For the quantum  $R$  matrix (3.4), one may check that the boundary  $K$  matrix  $K_-(u)$  given by

$$K_-(u) = \begin{pmatrix} A_-(u) & B_-(u) & 0 \\ C_-(u) & D_-(u) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.5)$$

with

$$\begin{aligned} A_-(u) &= \frac{2q^{2u+4} - q^{2u+2} + q^{2u} - 2q^{u+c_a+2} - 2q^{u-c_a+2} + q^2 + 1 + (q^2 - 1)(q^{2u} - 1)\sigma_a^z}{2(q^{-u+c_a+2} - 1)(q^{u-c_a+2} - 1)} \\ B_-(u) &= \frac{q(q^2 - 1)(q^{2u} - 1)\sigma_a^-}{(q^{-u+c_a+2} - 1)(q^{u-c_a+2} - 1)} \\ C_-(u) &= \frac{q(q^2 - 1)(q^{2u} - 1)\sigma_a^+}{(q^{-u+c_a+2} - 1)(q^{u-c_a+2} - 1)} \\ D_-(u) &= \frac{q^{2u+4} + q^{2u+2} - 2q^{u+c_a+2} - 2q^{u-c_a+2} + q^4 - q^2 + 2 - q^2(q^2 - 1)(q^{2u} - 1)\sigma_a^z}{2(q^{-u+c_a+2} - 1)(q^{u-c_a+2} - 1)} \end{aligned}$$

satisfies the graded RE (2.8) (see the appendix). Then the boundary  $K$  matrix  $K_+(u)$  defined by

$$K_+(u) = \begin{pmatrix} A_+(u) & B_+(u) & 0 \\ C_+(u) & D_+(u) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.6)$$

with

$$\begin{aligned} A_+(u) &= -[q^u(q^{2u+c_b+4} + 2q^{2u+c_b} - 2q^{u+2} - 2q^{u+2c_b} + q^{c_b} - q^{2u+c_b+2} + q^{c_b+2}) \\ &\quad + q^{u+c_b}(q^{2u+2} - 1)(q^2 - 1)\sigma_b^z][2(q^{u+4} - q^{c_b})(q^u - q^{c_b})]^{-1} \\ B_+(u) &= \frac{q^{u+c_b}(q^{2u+2} - 1)(q^2 - 1)\sigma_b^-}{(q^{u+4} - q^{c_b})(q^u - q^{c_b})} \\ C_+(u) &= \frac{q^{u+c_b}(q^{2u+2} - 1)(q^2 - 1)\sigma_b^+}{(q^{u+4} - q^{c_b})(q^u - q^{c_b})} \\ D_+(u) &= -[q^u(q^{2u+c_b+4} + q^{2u+c_b+2} - 2q^{u+4} - 2q^{u+2c_b+2} + 2q^{c_b+4} - q^{c_b+2} + q^{c_b}) \\ &\quad - q^{u+c_b}(q^{2u+2} - 1)(q^2 - 1)\sigma_b^z][2(q^{u+4} - q^{c_b})(q^u - q^{c_b})]^{-1} \end{aligned}$$

is a solution of the graded RE (2.11).

It can be shown that the Hamiltonian (3.2) is related to the derivative of the corresponding boundary transfer matrix  $\tau(u)$  with respect to the spectral parameter  $u$  at  $u = 0$  (up to an unimportant additive chemical potential term)

$$-H = \sum_{j=1}^{L-1} H_{j,j+1} + \frac{1}{2} K'_-(0) + \frac{\text{str}_0(K_+(0)H_{L0})}{\text{str}_0 K_+(0)} \quad (3.7)$$



with

$$H_{i,j} = \frac{d}{du} P_{i,j} R_{i,j}(u) |_{u=0} = P_{i,j} R'_{i,j}(0)$$

where  $P$  is a  $Z_2$ -graded permutation operator with the grading  $P[1] = P[2] = 1$  and  $P[3] = 0$ .

The second choice of integrable couplings (3.3) results from use of an  $R$  matrix obtained by imposing  $Z_2$ -grading to the fundamental  $q$ -deformed  $U_q(su(3))$   $R$  matrix which gives

$$R(u) = \begin{pmatrix} -q^u + q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q(q^u - 1) & 0 & -q^u(1 - q^2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q(q^u - 1) & 0 & 0 & q^u(1 - q^2) & 0 & 0 & 0 \\ 0 & -(1 - q^2) & 0 & -q(q^u - 1) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q^u + q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q(q^u - 1) & 0 & q^u(1 - q^2) & 0 \\ 0 & 0 & 1 - q^2 & 0 & 0 & 0 & q(q^u - 1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 - q^2 & 0 & q(q^u - 1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^u - q^2 \end{pmatrix} \tag{3.8}$$

where again  $u$  is the spectral parameter and we adopt the same choice for the  $Z_2$ -grading of the basis states as before.

We now solve (2.8) and (2.11) for  $K_-(u)$  and  $K_+(u)$ . For the quantum  $R$ -matrix (3.8), one may check that the matrix  $K_-(u)$  given by (3.5) with

$$A_-(u) = \frac{q^{2u+4} - q^{2u+2} + 2q^{2u} - 2q^{u+c_a+2} - 2q^{u-c_a+2} + q^4 + q^2 - q^2(q^2 - 1)(q^{2u} - 1)\sigma_a^z}{2(q^{-u-c_a+2} - 1)(q^{u+c_a+2} - 1)}$$

$$B_-(u) = -\frac{q(q^2 - 1)(q^{2u} - 1)\sigma_a^-}{(q^{-u-c_a+2} - 1)(q^{u+c_a+2} - 1)}$$

$$C_-(u) = -\frac{q(q^2 - 1)(q^{2u} - 1)\sigma_a^+}{(q^{-u-c_a+2} - 1)(q^{u+c_a+2} - 1)}$$

$$D_-(u) = \frac{q^{2u+2} + q^{2u} - 2q^{u+c_a+2} - 2q^{u-c_a+2} + 2q^4 - q^2 + 1 + (q^2 - 1)(q^{2u} - 1)\sigma_a^z}{2(q^{-u-c_a+2} - 1)(q^{u+c_a+2} - 1)}$$

satisfies the graded RE (2.8). For this case  $K_+(u)$  defined by (3.6) with

$$A_+(u) = [q^u(2q^{2u+c_b+4} + q^{2u+c_b} - 2q^{u+4} - 2q^{u+2c_b+6} + q^{c_b+8} - q^{2u+c_b+2} + q^{c_b+6}) - q^{u+c_b}(q^{2u} - q^6)(q^2 - 1)\sigma_b^z] \{2q^2(q^{u+c_b} - 1)(q^{u+c_b} - q^4)\}^{-1}$$

$$B_+(u) = -\frac{q^{u+c_b}(q^{2u} - q^6)(q^2 - 1)\sigma_b^-}{q^2(q^{u+c_b} - 1)(q^{u+c_b} - q^4)}$$

$$C_+(u) = -\frac{q^{u+c_b}(q^{2u} - q^6)(q^2 - 1)\sigma_b^+}{q^2(q^{u+c_b} - 1)(q^{u+c_b} - q^4)}$$

$$D_+(u) = [q^u(q^{2u+c_b+2} + q^{2u+c_b} - 2q^{u+2} - 2q^{u+2c_b+4} + 2q^{c_b+4} + q^{c_b+8} - q^{c_b+6}) + q^{u+c_b}(q^{2u} - q^6)(q^2 - 1)\sigma_b^z] \{2q^2(q^{u+c_b} - 1)(q^{u+c_b} - q^4)\}^{-1}$$

is a solution of the graded RE (2.11).

It can be shown for this case also that the Hamiltonian (3.3) can be embedded into the boundary transfer matrix  $\tau(u)$  with respect to the spectral parameter  $u$  at  $u = 0$  by (3.7).

#### 4. The Bethe ansatz solutions

Having established the quantum integrability of the models, let us now diagonalize the Hamiltonian (3.2) by means of the algebraic Bethe ansatz method [31, 32]. We introduce the ‘doubled’ monodromy matrix  $\mathcal{T}(u)$ :

$$\mathcal{T}(u) = T(u)K_-(u)\tilde{T}(u) \equiv \begin{pmatrix} \mathcal{A}_{11}(u) & \mathcal{A}_{12}(u) & \mathcal{B}_1(u) \\ \mathcal{A}_{21}(u) & \mathcal{A}_{22}(u) & \mathcal{B}_2(u) \\ \mathcal{C}_1(u) & \mathcal{C}_2(u) & \mathcal{D}(u) \end{pmatrix} \tag{4.1}$$

where  $\tilde{T}(u) = T^{-1}(-u)$ . Substituting (4.1) into equation (2.3), we may draw the following commutation relations:

$$\begin{aligned} \check{A}_{bd}(u_1)C_c(u_2) &= \frac{(q^{u_1-u_2+2}-1)(q^{u_1+u_2}-1)}{(q^{u_1-u_2}-1)(q^{u_1+u_2}-q^2)}r(u_1+u_2-2)_{gh}^{eb}r(u_1-u_2)_{cd}^{ih}C_e(u_2)\check{A}_{gi}(u_1) \\ &\quad - \frac{q^{u_1+u_2+1}(1-q^2)(q^{2u_1}-1)(q^{2u_2}-1)}{(q^{u_1+u_2}-q^2)(q^{2u_1}-q^2)(q^{2u_2}-q^2)}r(2u_1-2)_{cd}^{gb}C_g(u_1)\mathcal{D}(u_2) \\ &\quad + \frac{q^{u_1-u_2}(1-q^2)(q^{2u_1}-1)}{(q^{u_1-u_2}-1)(q^{2u_1}-q^2)}r(2u_1-2)_{id}^{gb}C_g(u_1)\check{A}_{ic}(u_2) \end{aligned} \quad (4.2)$$

$$\begin{aligned} \mathcal{D}(u_1)C_b(u_2) &= \frac{(q^{u_1-u_2+2}-1)(q^{u_1+u_2}-1)}{(q^{u_1-u_2}-1)(q^{u_1+u_2}-q^2)}C_b(u_2)\mathcal{D}(u_1) \\ &\quad - \frac{(1-q^2)(q^{2u_2}-1)}{(q^{2u_2}-q^2)(q^{u_1-u_2}-1)}C_b(u_1)\mathcal{D}(u_2) - \frac{1-q^2}{q^{u_1+u_2}-q^2}C_d(u_1)\check{A}_{db}(u_2). \end{aligned} \quad (4.3)$$

Here

$$\mathcal{A}_{bd}(u) = \check{A}_{bd}(u) + \frac{q^{2u}(1-q^2)}{q^{2u}-q^2}\delta_{bd}\mathcal{D}(u) \quad (4.4)$$

and the matrix  $r(u)$ , which in turn satisfies the quantum Yang–Baxter equation

$$r_{12}(u_1-u_2)r_{13}(u_1)r_{23}(u_2) = r_{23}(u_2)r_{13}(u_1)r_{12}(u_1-u_2) \quad (4.5)$$

takes the form

$$\begin{aligned} r_{bb}^{bb}(u) &= 1 & r_{12}^{12}(u) &= -\frac{1-q^2}{q^{u+2}-1} & r_{21}^{21}(u) &= -\frac{q^u(1-q^2)}{q^{u+2}-1} \\ r_{db}^{bd}(u) &= \frac{q(q^u-1)}{q^{u+2}-1} & (b \neq d, b, d = 1, 2). \end{aligned}$$

Choosing the Bethe state  $|\Omega\rangle$  as

$$|\Omega\rangle = C_{d_1}(u_1) \cdots C_{d_N}(u_N)|\Psi\rangle F^{d_1 \cdots d_N}$$

with  $|\Psi\rangle$  being the pseudovacuum, the indices  $d_j$  run over the values 1, 2, and  $F^{d_1 \cdots d_N}$  is a function of the spectral parameters  $u_j$ . Applying the boundary transfer matrix  $\tau(u)$  to the state  $|\Omega\rangle$ , we have  $\tau(u)|\Omega\rangle = \Lambda(u)|\Omega\rangle$ , with the eigenvalue

$$\begin{aligned} \Lambda(u) &= \frac{q^{2u+2}-1}{q^{2u}-q^2} \frac{q^{u+2}-q^{c_b}}{q^u-q^{c_b}} \frac{q^u-q^{c_b+2}}{q^{u+4}-q^{c_b}} \left( -\frac{q^u(q^u-q^2)}{q^{u+2}-1} \right)^L \\ &\quad \times \prod_{j=1}^N \frac{q^2(q^{u+u_j}-1)(q^{u-u_j+2}-1)}{(q^{u+u_j}-q^2)(q^{u-u_j+2}-q^2)} \\ &\quad - \frac{q^2(q^{2u}-1)}{q^{2u}-q^2} \left( \frac{q^2(q^u-1)^2}{(q^{u+2}-1)(q^{-u+2}-1)} \right)^L \\ &\quad \times \prod_{j=1}^N \frac{q^2(q^{u+u_j}-1)(q^{u-u_j+2}-1)}{(q^{u+u_j}-q^2)(q^{u-u_j+2}-q^2)} \Lambda^{(1)}(u; \{u_j\}) \end{aligned}$$

provided the parameters  $\{u_j\}$  satisfy

$$\frac{q^{2u_j+2}-1}{q^2(q^{2u_j}-1)} \frac{(q^{u_j+2}-q^{c_b})}{(q^{u_j}-q^{c_b})} \frac{(q^{u_j}-q^{c_b+2})}{(q^{u_j+4}-q^{c_b})} \left( \frac{q^{u_j}-q^2}{q(q^{u_j}-1)} \right)^{2L} = \Lambda^{(1)}(u_j; \{u_i\}). \quad (4.6)$$

Here  $\Lambda^{(1)}(u; \{u_j\})$  is the eigenvalue of the nested boundary transfer matrix  $\tau^{(1)}(u)$ :

$$\tau^{(1)}(u) = \text{str}(K_+^{(1)}(u)T^{(1)}(u, \{u_j\})K_-^{(1)}(u)T^{(1)-1}(-u, \{u_j\})) \quad (4.7)$$

with the fermionic and fermionic (FF) grading that means  $[[1]] = [[2]] = 1$ . It arises out of the  $r$  matrices from the first term on the right-hand side of (4.2). We can prove that the nested boundary  $K$  matrices  $K_{\pm}^{(1)}(u)$  are

$$K_{-}^{(1)}(u) = \begin{pmatrix} A_{-}^{(1)}(u) & B_{-}^{(1)}(u) \\ C_{-}^{(1)}(u) & D_{-}^{(1)}(u) \end{pmatrix} \quad (4.8)$$

with

$$\begin{aligned} A_{-}^{(1)}(u) &= \frac{2q^{2u+2} - q^{2u} + q^{2u-2} - 2q^{u+c_a} - 2q^{u-c_a+2} + q^2 + 1 + (q^2 - 1)(q^{2u-2} - 1)\sigma_a^z}{2(q^{-u+c_a+2} - 1)(q^{u-c_a+2} - 1)} \\ B_{-}^{(1)}(u) &= \frac{q(q^2 - 1)(q^{2u-2} - 1)\sigma_a^-}{(q^{-u+c_a+2} - 1)(q^{u-c_a+2} - 1)} \\ C_{-}^{(1)}(u) &= \frac{q(q^2 - 1)(q^{2u-2} - 1)\sigma_a^+}{(q^{-u+c_a+2} - 1)(q^{u-c_a+2} - 1)} \\ D_{-}^{(1)}(u) &= \frac{q^{2u+2} + q^{2u} - 2q^{u+c_a} - 2q^{u-c_a+2} + q^4 - q^2 + 2 - q^2(q^2 - 1)(q^{2u-2} - 1)\sigma_a^z}{2(q^{-u+c_a+2} - 1)(q^{u-c_a+2} - 1)} \end{aligned}$$

satisfying the nested graded RE

$$r_{12}(u_1 - u_2) K_{-}^{(1)}(u_1) r_{21}(u_1 + u_2) K_{-}^{(1)}(u_2) = K_{-}^{(1)}(u_2) r_{12}(u_1 + u_2) K_{-}^{(1)}(u_2) r_{21}(u_1 - u_2) \quad (4.9)$$

and

$$K_{+}^{(1)}(u) = \begin{pmatrix} A_{+}^{(1)}(u) & B_{+}^{(1)}(u) \\ C_{+}^{(1)}(u) & D_{+}^{(1)}(u) \end{pmatrix} \quad (4.10)$$

with

$$\begin{aligned} A_{+}^{(1)}(u) &= -[q^u(q^{2u+c_b+4} + 2q^{2u+c_b} - 2q^{u+2} - 2q^{u+2c_b} + q^{c_b} - q^{2u+c_b+2} + q^{c_b+2}) \\ &\quad + q^{u+c_b}(q^{2u+2} - 1)(q^2 - 1)\sigma_b^z][2(q^{u+4} - q^{c_b})(q^u - q^{c_b})]^{-1} \\ B_{+}^{(1)}(u) &= \frac{q^{u+c_b}(q^{2u+2} - 1)(q^2 - 1)\sigma_b^-}{(q^{u+4} - q^{c_b})(q^u - q^{c_b})} \\ C_{+}^{(1)}(u) &= \frac{q^{u+c_b}(q^{2u+2} - 1)(q^2 - 1)\sigma_b^+}{(q^{u+4} - q^{c_b})(q^u - q^{c_b})} \\ D_{+}^{(1)}(u) &= -[q^u(q^{2u+c_b+4} + q^{2u+c_b+2} - 2q^{u+4} - 2q^{u+2c_b+2} + 2q^{c_b+4} - q^{c_b+2} + q^{c_b}) \\ &\quad - q^{u+c_b}(q^{2u+2} - 1)(q^2 - 1)\sigma_b^z][2(q^{u+4} - q^{c_b})(q^u - q^{c_b})]^{-1} \end{aligned}$$

satisfying the nested graded RE

$$\begin{aligned} r_{21}^{st_1 ist_2}(-u_1 + u_2) K_{+}^{(1)st_1}(u_1) \{ [r_{21}^{st_1}(u_1 + u_2)]^{-1} \}^{ist_2} K_{+}^{(1)ist_2}(u_2) \\ = K_{+}^{(1)ist_2}(u_2) \{ [r_{12}^{ist_2}(u_1 + u_2)]^{-1} \}^{st_1} K_{+}^{(1)st_1}(u_1) r_{12}^{st_1 ist_2}(-u_1 + u_2). \end{aligned} \quad (4.11)$$

For the one-dimensional  $q$ -deformed supersymmetric  $U_q(gl(2|1))$   $t$ - $J$  model  $R$  matrix (3.4), choosing the pseudovacuum  $|\Psi\rangle = (0, 0, 1)^T$ , then gives

$$\begin{aligned} T_{dd}(u)|\Psi\rangle &= (q(q^u - 1))^L |\Psi\rangle & T_{33}(u)|\Psi\rangle &= (q^u - q^2)^L |\Psi\rangle \\ T_{3d}(u)|\Psi\rangle &\neq 0 & T_{db}(u)|\Psi\rangle &= 0 & T_{d3}(u)|\Psi\rangle &= 0 \\ \tilde{T}_{dd}(u)|\Psi\rangle &= \left( -\frac{q^{u+1}(q^u - 1)}{(q^u - q^2)(q^{u+2} - 1)} \right)^L |\Psi\rangle & \tilde{T}_{33}(u)|\Psi\rangle &= \left( -\frac{q^u}{q^{u+2} - 1} \right)^L |\Psi\rangle \\ \tilde{T}_{3d}(u)|\Psi\rangle &\neq 0 & \tilde{T}_{db}(u)|\Psi\rangle &= 0 & \tilde{T}_{d3}(u)|\Psi\rangle &= 0 \end{aligned} \quad (4.12)$$

where  $d \neq b$ ,  $d, b = 1, 2$ . We also find

$$\begin{aligned} T_{\alpha 3}(u) \tilde{T}_{3\alpha}(u) |\Psi\rangle &= \frac{q^{2u}(1-q^2)}{q^{2u}-q^2} [\tilde{T}_{33}(u) T_{33}(u) - T_{\alpha\alpha} \tilde{T}_{\alpha\alpha}(u)] |\Psi\rangle \\ T_{\alpha 3}(u) \tilde{T}_{3\beta}(u) |\Psi\rangle &= 0 \quad \alpha \neq \beta. \end{aligned} \quad (4.13)$$

This leads to

$$\begin{aligned} \mathcal{D}(u) |\Psi\rangle &= T_{3\alpha}(u) K_{-\alpha\beta}(u) \tilde{T}_{\beta 3}(u) |\Psi\rangle = \left( -\frac{q^u(q^u - q^2)}{q^{u+2} - 1} \right)^L |\Psi\rangle \\ \mathcal{B}_d(u) |\Psi\rangle &= T_{d\alpha}(u) K_{-\alpha\beta}(u) \tilde{T}_{\beta 3}(u) |\Psi\rangle = 0 \\ \mathcal{C}_d(u) |\Psi\rangle &\neq 0 \\ \check{\mathcal{A}}_{db}(u) |\Psi\rangle &= T_{d\alpha}(u) K_{-\alpha\beta}(u) \tilde{T}_{\beta b}(u) |\Psi\rangle = \left( \frac{q^2(q^u - 1)^2}{(q^{u+2} - 1)(q^{-u+2} - 1)} \right)^L K_{-(u)db} |\Psi\rangle \\ \check{\mathcal{A}}_{dd}(u) |\Psi\rangle &= T_{dd}(u) \left( K_{-(u)dd} - \frac{q^{2u}(1-q^2)}{q^{2u}-q^2} \right) \tilde{T}_{dd}(u) |\Psi\rangle \\ &\quad + \frac{q^{2u}(1-q^2)}{q^{2u}-q^2} \left( -\frac{q^u(q^u - q^2)}{q^{u+2} - 1} \right)^L |\Psi\rangle. \end{aligned} \quad (4.14)$$

Here

$$K_{-(u)d\alpha} - \frac{q^{2u}(1-q^2)}{q^{2u}-q^2} = \frac{q^2(q^{2u}-1)}{q^{2u}-q^2} K_{-(u-1)d\alpha} \quad \alpha = d, b$$

satisfy equation (2.8) for the reduced problem. By (4.4), we have

$$\begin{aligned} \check{\mathcal{A}}_{dd}(u) |\Psi\rangle &= \frac{q^2(q^{2u}-1)}{q^{2u}-q^2} K_{-(u)dd}^{(1)} |\Psi\rangle \\ &= \left( K_{-(u)dd} - \frac{q^{2u}(1-q^2)}{q^{2u}-q^2} \right) \left( \frac{q^2(q^u-1)^2}{(q^{u+2}-1)(q^{-u+2}-1)} \right)^L |\Psi\rangle \\ \check{\mathcal{A}}_{db}(u) |\Psi\rangle &= \frac{q^2(q^{2u}-1)}{q^{2u}-q^2} K_{-(u)db}^{(1)} = K_{-(u)db} \left( \frac{q^2(q^u-1)^2}{(q^{u+2}-1)(q^{-u+2}-1)} \right)^L |\Psi\rangle. \end{aligned} \quad (4.15)$$

In our calculation, use of the following relations has also been made:

$$\begin{aligned} q^{2u}(1-q^2)T_{11}(u)\tilde{T}_{11}(u) + q^{2u}(1-q^2)T_{12}(u)\tilde{T}_{21}(u) + (q^{2u}-q^2)T_{13}(u)\tilde{T}_{31}(u) \\ = -(q^{2u+2}-1)\tilde{T}_{31}(u)T_{13}(u) + q^{2u}(1-q^2)\tilde{T}_{32}(u)T_{23}(u) \\ + q^{2u}(1-q^2)\tilde{T}_{33}(u)T_{33}(u) \\ q^{2u}(1-q^2)T_{11}(u)\tilde{T}_{12}(u) + q^{2u}(1-q^2)T_{12}(u)\tilde{T}_{22}(u) + (q^{2u}-q^2)T_{13}(u)\tilde{T}_{32}(u) \\ = -q(q^{2u}-1)\tilde{T}_{32}(u)T_{13}(u) \\ q^{2u}(1-q^2)T_{21}(u)\tilde{T}_{11}(u) + q^{2u}(1-q^2)T_{22}(u)\tilde{T}_{21}(u) + (q^{2u}-q^2)T_{23}(u)\tilde{T}_{31}(u) \\ = -q(q^{2u}-1)\tilde{T}_{31}(u)T_{23}(u) \\ q^{2u}(1-q^2)T_{21}(u)\tilde{T}_{12}(u) + q^{2u}(1-q^2)T_{22}(u)\tilde{T}_{22}(u) + (q^{2u}-q^2)T_{23}(u)\tilde{T}_{32}(u) \\ = (1-q^2)\tilde{T}_{31}(u)T_{13}(u) - (q^{2u+2}-1)\tilde{T}_{32}(u)T_{23}(u) + q^{2u}(1-q^2)\tilde{T}_{33}(u)T_{33}(u) \end{aligned}$$

which come from a variant of the (graded) Yang–Baxter algebra

$$\overset{1}{T}(u)R(2u)\overset{2}{T}(u) = \overset{2}{T}(u)R(2u)\overset{1}{T}(u). \quad (4.16)$$

Implementing the change  $u \rightarrow u+1$  with respect to the original problem, one may check that the nested boundary  $K$  matrices (4.8) and (4.10) still satisfy the reflection equations (4.9) and (4.11) for the reduced problem. After some algebra, the nested boundary transfer matrix

$\tau^{(1)}(u)$  may be recognized as that for the  $(N+2)$ -site  $XXZ$  spin- $\frac{1}{2}$  open chain, which may be diagonalized following [31]. Here we merely give the final result:

$$\begin{aligned} \Lambda^{(1)}(u; \{u_j\}) &= \frac{(q^{u+2} - q^{c_b}) (q^u - q^{c_b+2})}{(q^u - q^{c_b}) (q^{u+4} - q^{c_b})} \prod_{\alpha=a,b} \frac{q^{u+c_\alpha+1} - 1}{q^{c_\alpha} (q^{u-c_\alpha-1} - q^2)} \\ &\times \left\{ \frac{q^{2u+2} - 1}{q^2 (q^{2u} - 1)} \prod_{k=1}^M \frac{(q^{u-u_k^{(1)}} - q^2)(q^{u+u_k^{(1)}} - q^2)}{q^2 (q^{u-u_k^{(1)}} - 1)(q^{u+u_k^{(1)}} - 1)} + \frac{q^{2u} - q^2}{q^2 (q^{2u} - 1)} \right. \\ &\times \prod_{\alpha=a,b} \frac{q^2 (q^{u+c_\alpha-1} - 1) (q^{u-c_\alpha-1} - 1)}{(q^{u+c_\alpha+1} - 1) (q^{u-c_\alpha+1} - 1)} \\ &\left. \times \prod_{j=1}^N \frac{(q^{u+u_j} - q^2)(q^{u-u_j+2} - q^2)}{q^2 (q^{u+u_j} - 1)(q^{u-u_j+2} - 1)} \prod_{k=1}^M \frac{q^2 (q^{u-u_k^{(1)+2}} - 1)(q^{u+u_k^{(1)+2}} - 1)}{(q^{u-u_k^{(1)+2}} - q^2)(q^{u+u_k^{(1)+2}} - q^2)} \right\} \end{aligned} \quad (4.17)$$

provided the parameters  $\{u_k^{(1)}\}$  satisfy

$$\begin{aligned} \prod_{\alpha=a,b} \frac{(q^{u_k^{(1)+c_\alpha+1} - q^2)(q^{u_k^{(1)-c_\alpha+1} - q^2)} \prod_{j=1}^N (q^{u_k^{(1)+u_j} - q^2)(q^{u_k^{(1)-u_j+2} - q^2)} \\ q^2 (q^{u_k^{(1)+c_\alpha+1} - 1)(q^{u_k^{(1)-c_\alpha+1} - 1)} \prod_{j=1}^N q^2 (q^{u_k^{(1)+u_j} - 1)(q^{u_k^{(1)-u_j+2} - 1)} \\ = \prod_{\substack{l=1 \\ l \neq k}}^M \frac{(q^{u_k^{(1)-u_l^{(1)+2} - q^4})(q^{u_k^{(1)+u_l^{(1)+2} - q^4})}{q^4 (q^{u_k^{(1)-u_l^{(1)+2} - 1})(q^{u_k^{(1)+u_l^{(1)+2} - 1})}. \end{aligned} \quad (4.18)$$

After a shift of the parameters  $u_j \rightarrow u_j + 1, u_k^{(1)} \rightarrow u_k^{(1)}$ , the Bethe ansatz equations (4.6) and (4.18) may be rewritten as follows:

$$\begin{aligned} \left( \frac{q(q^{u_j-1} - 1)}{q^{u_j+1} - 1} \right)^{2L} \prod_{\alpha=a,b} \frac{q^{c_\alpha+2} (q^{u_j-c_\alpha-2} - 1)}{q^{u_j+c_\alpha+2} - 1} = \prod_{k=1}^M \frac{q^2 (q^{u_j-u_k^{(1)}-1} - 1) (q^{u_j+u_k^{(1)}-1} - 1)}{(q^{u_j-u_k^{(1)+1} - 1}) (q^{u_j+u_k^{(1)+1} - 1)} \\ \prod_{\alpha=a,b} \frac{q^2 (q^{u_k^{(1)-c_\alpha-1} - 1)(q^{u_k^{(1)+c_\alpha-1} - 1)} \prod_{j=1}^N q^2 (q^{u_k^{(1)-u_j-1} - 1)(q^{u_k^{(1)+u_j-1} - 1)} \\ (q^{u_k^{(1)-c_\alpha+1} - 1)(q^{u_k^{(1)+c_\alpha+1} - 1)} \prod_{j=1}^N (q^{u_k^{(1)-u_j+1} - 1)(q^{u_k^{(1)+u_j+1} - 1)} \\ = \prod_{\substack{l=1 \\ l \neq k}}^M \frac{q^4 (q^{u_k^{(1)-u_l^{(1)}-2} - 1)(q^{u_k^{(1)+u_l^{(1)}-2} - 1)}{(q^{u_k^{(1)-u_l^{(1)+2} - 1})(q^{u_k^{(1)+u_l^{(1)+2} - 1})} \end{aligned} \quad (4.19)$$

or

$$\begin{aligned} \left( \frac{\sinh \gamma(u_j - 1)}{\sinh \gamma(u_j + 1)} \right)^{2L} \prod_{\alpha=a,b} \frac{\sinh \gamma(u_j - c_\alpha - 2)}{\sinh \gamma(u_j + c_\alpha + 2)} \\ = \prod_{k=1}^M \frac{\sinh \gamma(u_j - u_k^{(1)} - 1) \sinh \gamma(u_j + u_k^{(1)} - 1)}{\sinh \gamma(u_j - u_k^{(1)} + 1) \sinh \gamma(u_j + u_k^{(1)} + 1)} \\ \prod_{\alpha=a,b} \frac{\sinh \gamma(u_k^{(1)} - c_\alpha - 1) \sinh \gamma(u_k^{(1)} + c_\alpha - 1)}{\sinh \gamma(u_k^{(1)} - c_\alpha + 1) \sinh \gamma(u_k^{(1)} + c_\alpha + 1)} \\ \times \prod_{j=1}^N \frac{\sinh \gamma(u_k^{(1)} - u_j - 1) \sinh \gamma(u_k^{(1)} + u_j - 1)}{\sinh \gamma(u_k^{(1)} - u_j + 1) \sinh \gamma(u_k^{(1)} + u_j + 1)} \\ = \prod_{\substack{l=1 \\ l \neq k}}^M \frac{\sinh \gamma(u_k^{(1)} - u_l^{(1)} - 2) \sinh \gamma(u_k^{(1)} + u_l^{(1)} - 2)}{\sinh \gamma(u_k^{(1)} - u_l^{(1)} + 2) \sinh \gamma(u_k^{(1)} + u_l^{(1)} + 2)} \end{aligned}$$

with  $q = \exp 2\gamma$ . The corresponding energy eigenvalue  $E$  of the model is

$$E = - \sum_{j=1}^N \frac{4}{\sinh \gamma(u_j - 1) \sinh \gamma(u_j + 1)}. \tag{4.20}$$

(modulo an unimportant additive constant, which we drop).

We now perform the algebraic Bethe ansatz procedure for the couplings (3.3). Again we introduce the ‘doubled’ monodromy matrix  $\mathcal{T}(u)$  as (4.1). Substituting (4.1) into equation (2.3), we now find the following commutation relations:

$$\begin{aligned} \check{A}_{bd}(u_1)C_c(u_2) &= \frac{(q^{u_1-u_2} - q^2)(q^{u_1+u_2} - q^4)}{q^2(q^{u_1-u_2} - 1)(q^{u_1+u_2} - q^2)} r(u_1 + u_2 - 2)_{gh}^{eb} r(u_1 - u_2)_{cd}^{ih} C_e(u_2) \check{A}_{gi}(u_1) \\ &+ \frac{q^{u_1+u_2}(1 - q^2)(q^{2u_1} - q^4)(q^{2u_2} - 1)}{(q^{u_1+u_2} - q^2)(q^{2u_1} - q^2)(q^{2u_2} - q^2)} r(2u_1 - 2)_{cd}^{gb} C_g(u_1) \mathcal{D}(u_2) \\ &- \frac{q^{u_1-u_2}(1 - q^2)(q^{2u_1} - q^4)}{q^2(q^{u_1-u_2} - 1)(q^{2u_1} - q^2)} r(2u_1 - 2)_{id}^{gb} C_g(u_1) \check{A}_{ic}(u_2) \end{aligned} \tag{4.21}$$

as well as (4.3) and (4.4). The matrix  $r(u)$ , which also satisfies the quantum Yang–Baxter equation (4.5), takes the form

$$\begin{aligned} r_{bb}^{bb}(u) &= 1 & r_{12}^{12}(u) &= \frac{1 - q^2}{q^u - q^2} & r_{21}^{21}(u) &= \frac{q^u(1 - q^2)}{q^u - q^2} \\ r_{db}^{bd}(u) &= \frac{q(q^u - 1)}{q^u - q^2} & (b \neq d, b, d = 1, 2). \end{aligned}$$

Acting the  $\tau(u)$  on the Bethe state  $|\Omega\rangle$ ,  $|\Omega\rangle = C_{d_1}(u_1) \cdots C_{d_N}(u_N) |\Psi\rangle F^{d_1 \cdots d_N}$ , we have  $\tau(u)|\Omega\rangle = \Lambda(u)|\Omega\rangle$ , with the eigenvalue

$$\begin{aligned} \Lambda(u) &= \frac{q^{2u} - q^4}{q^{2u} - 1} \frac{q^{u+c_b} - 1}{q^{u+c_b-2} - 1} \frac{q^{u+c_b-2} - q^2}{q^{u+c_b-2} - q^4} \left( - \frac{q^u(q^u - q^2)}{q^{u+2} - 1} \right)^L \\ &\times \prod_{j=1}^N \frac{q^2(q^{u+u_j} - 1)(q^{u-u_j+2} - 1)}{(q^{u+u_j} - q^2)(q^{u-u_j+2} - q^2)} \\ &- \frac{q^2(q^{2u} - 1)}{q^{2u} - q^2} \left( \frac{q^2(q^u - 1)^2}{(q^{u+2} - 1)(q^{-u+2} - 1)} \right)^L \\ &\times \prod_{j=1}^N \frac{(q^{u+u_j} - q^4)(q^{u-u_j+2} - q^4)}{q^2(q^{u+u_j} - q^2)(q^{u-u_j+2} - q^2)} \Lambda^{(1)}(u; \{u_j\}) \end{aligned}$$

provided the parameters  $\{u_j\}$  satisfy

$$\begin{aligned} \frac{q^{2u_j} - q^2}{q^{2u_j} - 1} \frac{(q^{u_j+c_b} - 1)}{(q^{u_j+c_b-2} - 1)} \frac{(q^{u_j+c_b-2} - q^2)}{(q^{u_j+c_b-2} - q^4)} \left( \frac{q^{u_j} - q^2}{q(q^{u_j} - 1)} \right)^{2L} \\ \times \prod_{\substack{i=1 \\ i \neq j}}^N \frac{q^4(q^{u_j+u_i} - 1)(q^{u_j-u_i+2} - 1)}{(q^{u_j+u_i} - q^4)(q^{u_j-u_i+2} - q^4)} = -\Lambda^{(1)}(u_j; \{u_i\}). \end{aligned} \tag{4.22}$$

Here  $\Lambda^{(1)}(u; \{u_i\})$  is the eigenvalue of the nested boundary transfer matrix  $\tau^{(1)}(u)$  (4.7), which arises out of the  $r$  matrices from the first term on the right-hand side of (4.21). We can prove

that the nested boundary  $K$  matrices  $K_-^{(1)}(u)$  (4.8), with

$$\begin{aligned} A_-^{(1)}(u) &= \frac{q^{2u+2} - q^{2u} + 2q^{2u-2} - 2q^{u+c_a+2} - 2q^{u-c_a} + q^4 + q^2 - q^2(q^2 - 1)(q^{2u-2} - 1)\sigma_a^z}{2(q^{-u-c_a+2} - 1)(q^{u+c_a+2} - 1)} \\ B_-^{(1)}(u) &= -\frac{q(q^2 - 1)(q^{2u-2} - 1)\sigma_a^-}{(q^{-u-c_a+2} - 1)(q^{u+c_a+2} - 1)} \\ C_-^{(1)}(u) &= -\frac{q(q^2 - 1)(q^{2u-2} - 1)\sigma_a^+}{(q^{-u-c_a+2} - 1)(q^{u+c_a+2} - 1)} \\ D_-^{(1)}(u) &= \frac{q^{2u} + q^{2u-2} - 2q^{u+c_a+2} - 2q^{u-c_a} + 2q^4 - q^2 + 1 + (q^2 - 1)(q^{2u-2} - 1)\sigma_a^z}{2(q^{-u-c_a+2} - 1)(q^{u+c_a+2} - 1)} \end{aligned}$$

satisfies the nested graded RE (4.9), and the nested boundary  $K$  matrix  $K_+^{(1)}(u)$  (4.10), with

$$\begin{aligned} A_+^{(1)}(u) &= [q^u(2q^{2u+c_b+4} + q^{2u+c_b} - 2q^{u+4} - 2q^{u+2c_b+6} + q^{c_b+8} - q^{2u+c_b+2} + q^{c_b+6}) \\ &\quad - q^{u+c_b}(q^{2u} - q^6)(q^2 - 1)\sigma_b^z][2q^2(q^{u+c_b} - 1)(q^{u+c_b} - q^4)]^{-1} \\ B_+^{(1)}(u) &= -\frac{q^{u+c_b}(q^{2u} - q^6)(q^2 - 1)\sigma_b^-}{q^2(q^{u+c_b} - 1)(q^{u+c_b} - q^4)} \\ C_+^{(1)}(u) &= -\frac{q^{u+c_b}(q^{2u} - q^6)(q^2 - 1)\sigma_b^+}{q^2(q^{u+c_b} - 1)(q^{u+c_b} - q^4)} \\ D_+^{(1)}(u) &= [q^u(q^{2u+c_b+2} + q^{2u+c_b} - 2q^{u+2} - 2q^{u+2c_b+4} + 2q^{c_b+4} + q^{c_b+8} - q^{c_b+6}) \\ &\quad + q^{u+c_b}(q^{2u} - q^6)(q^2 - 1)\sigma_b^z][2q^2(q^{u+c_b} - 1)(q^{u+c_b} - q^4)]^{-1} \end{aligned}$$

satisfies the nested graded RE (4.11). For the one-dimensional  $q$ -deformed  $U_q(\mathfrak{su}(3))$   $t$ - $J$  model  $R$  matrix (3.8) with pseudovacuum  $|\Psi\rangle = (0, 0, 1)^T$  we have the same relations (4.12)–(4.14) holding true. Now

$$K_-(u)_{d\alpha} - \frac{q^{2u}(1 - q^2)}{q^{2u} - q^2} = \frac{q^2(q^{2u} - 1)}{q^{2u} - q^2} K_-(u - 1)_{d\alpha} \quad \alpha = d, b$$

satisfy the graded RE (2.8) for the reduced problem. By (4.4), we have (4.15). For this calculation, use of the following relations has also been made:

$$\begin{aligned} &q^{2u}(1 - q^2)T_{11}(u)\tilde{T}_{11}(u) + q^{2u}(1 - q^2)T_{12}(u)\tilde{T}_{21}(u) + (q^{2u} - q^2)T_{13}(u)\tilde{T}_{31}(u) \\ &\quad = (q^{2u} - q^2)\tilde{T}_{31}(u)T_{13}(u) + q^{2u}(1 - q^2)\tilde{T}_{32}(u)T_{23}(u) \\ &\quad\quad + q^{2u}(1 - q^2)\tilde{T}_{33}(u)T_{33}(u) \\ &q^{2u}(1 - q^2)T_{11}(u)\tilde{T}_{12}(u) + q^{2u}(1 - q^2)T_{12}(u)\tilde{T}_{22}(u) + (q^{2u} - q^2)T_{13}(u)\tilde{T}_{32}(u) \\ &\quad = q(q^{2u} - 1)\tilde{T}_{32}(u)T_{13}(u) \\ &q^{2u}(1 - q^2)T_{21}(u)\tilde{T}_{11}(u) + q^{2u}(1 - q^2)T_{22}(u)\tilde{T}_{21}(u) + (q^{2u} - q^2)T_{23}(u)\tilde{T}_{31}(u) \\ &\quad = q(q^{2u} - 1)\tilde{T}_{31}(u)T_{23}(u) \\ &q^{2u}(1 - q^2)T_{21}(u)\tilde{T}_{12}(u) + q^{2u}(1 - q^2)T_{22}(u)\tilde{T}_{22}(u) + (q^{2u} - q^2)T_{23}(u)\tilde{T}_{32}(u) \\ &\quad = (1 - q^2)\tilde{T}_{31}(u)T_{13}(u) + (q^{2u} - q^2)\tilde{T}_{32}(u)T_{23}(u) + q^{2u}(1 - q^2)\tilde{T}_{33}(u)T_{33}(u) \end{aligned} \tag{4.23}$$

which as before come from a variant of the graded Yang–Baxter algebra (4.16).

Implementing the change  $u \rightarrow u + 1$  with respect to the original problem, one may check that the boundary  $K$  matrices (4.8) and (4.10) still satisfy the reflection equations (4.9) and (4.11) for the reduced problem. After some algebra, the nested boundary transfer matrix  $\tau^{(1)}(u)$  may be recognized as that for the  $(N + 2)$ -site  $XXZ$  spin- $\frac{1}{2}$  open chain, which may be diagonalized following [31]. Again we present only the final result

$$\begin{aligned} \Lambda^{(1)}(u; \{u_j\}) &= -\frac{(q^{u+c_b} - 1)}{(q^{u+c_b-2} - 1)} \frac{(q^{u+c_b-2} - q^2)}{(q^{u+c_b-2} - q^4)} \prod_{\alpha=a,b} \frac{q^{c_\alpha} (q^{u-c_\alpha-1} - q^2)}{q^{u+c_\alpha+1} - 1} \\ &\times \left\{ \frac{q^{2u} - q^2}{q^{2u} - 1} \prod_{k=1}^M \frac{q^2 (q^{u-u_k^{(1)}+2} - 1) (q^{u+u_k^{(1)}-2} - 1)}{(q^{u-u_k^{(1)}+2} - q^2) (q^{u+u_k^{(1)}-2} - q^2)} + \frac{q^{2u+2} - 1}{q^{2u} - 1} \right. \\ &\times \prod_{\alpha=a,b} \frac{q^4 (q^{u-c_\alpha-1} - 1) (q^{u+c_\alpha-1} - 1)}{(q^{u-c_\alpha-1} - q^2) (q^{u+c_\alpha-1} - q^2)} \\ &\left. \times \prod_{j=1}^N \frac{q^2 (q^{u+u_j-2} - 1) (q^{u-u_j} - 1)}{(q^{u+u_j-2} - q^2) (q^{u-u_j} - q^2)} \prod_{k=1}^M \frac{(q^{u-u_k^{(1)}} - q^2) (q^{u+u_k^{(1)}-4} - q^2)}{q^2 (q^{u-u_k^{(1)}} - 1) (q^{u+u_k^{(1)}-4} - 1)} \right\} \end{aligned}$$

provided the parameters  $\{u_k^{(1)}\}$  satisfy

$$\begin{aligned} &\prod_{\alpha=a,b} \frac{q^4 (q^{u_k^{(1)}-c_\alpha-1} - 1) (q^{u_k^{(1)}+c_\alpha-1} - 1)}{(q^{u_k^{(1)}-c_\alpha-1} - q^2) (q^{u_k^{(1)}+c_\alpha-1} - q^2)} \prod_{j=1}^N \frac{q^2 (q^{u_k^{(1)}+u_j-2} - 1) (q^{u_k^{(1)}-u_j} - 1)}{(q^{u_k^{(1)}+u_j-2} - q^2) (q^{u_k^{(1)}-u_j} - q^2)} \\ &= \prod_{\substack{i=1 \\ l \neq k}}^M \frac{q^4 (q^{u_k^{(1)}-u_l^{(1)}+2} - 1) (q^{u_k^{(1)}+u_l^{(1)}-2} - 1)}{(q^{u_k^{(1)}-u_l^{(1)}+2} - q^4) (q^{u_k^{(1)}+u_l^{(1)}-2} - q^4)}. \end{aligned} \quad (4.24)$$

After a shift of the parameters  $u_j \rightarrow u_j + 1$ ,  $u_k^{(1)} \rightarrow u_k^{(1)} + 2$ , the Bethe ansatz equations (4.22) and (4.24) may be rewritten as follows:

$$\begin{aligned} &\left( \frac{q(q^{u_j-1} - 1)}{q^{u_j+1} - 1} \right)^{2L} \prod_{\alpha=a,b} \frac{q^{u_j+c_\alpha+2} - 1}{q^{c_\alpha+2} (q^{u_j-c_\alpha-2} - 1)} \prod_{\substack{i=1 \\ i \neq j}}^N \frac{(q^{u_j+u_i+2} - 1) (q^{u_j-u_i+2} - 1)}{q^4 (q^{u_j+u_i-2} - 1) (q^{u_j-u_i-2} - 1)} \\ &= \prod_{k=1}^M \frac{(q^{u_j-u_k^{(1)}+1} - 1) (q^{u_j+u_k^{(1)}+1} - 1)}{q^2 (q^{u_j-u_k^{(1)}-1} - 1) (q^{u_j+u_k^{(1)}-1} - 1)} \\ &\prod_{\alpha=a,b} \frac{(q^{u_k^{(1)}-c_\alpha+1} - 1) (q^{u_k^{(1)}+c_\alpha+1} - 1)}{(q^{u_k^{(1)}-c_\alpha-1} - 1) (q^{u_k^{(1)}+c_\alpha-1} - 1)} \prod_{j=1}^N \frac{(q^{u_k^{(1)}-u_j+1} - 1) (q^{u_k^{(1)}+u_j+1} - 1)}{q^2 (q^{u_k^{(1)}-u_j-1} - 1) (q^{u_k^{(1)}+u_j-1} - 1)} \\ &= \prod_{\substack{i=1 \\ l \neq k}}^M \frac{(q^{u_k^{(1)}-u_l^{(1)}+2} - 1) (q^{u_k^{(1)}+u_l^{(1)}+2} - 1)}{q^4 (q^{u_k^{(1)}-u_l^{(1)}-2} - 1) (q^{u_k^{(1)}+u_l^{(1)}-2} - 1)} \end{aligned} \quad (4.25)$$

or

$$\begin{aligned} &\left( \frac{\sinh \gamma(u_j - 1)}{\sinh \gamma(u_j + 1)} \right)^{2L} \prod_{\alpha=a,b} \frac{\sinh \gamma(u_j + c_\alpha + 2)}{\sinh \gamma(u_j - c_\alpha - 2)} \prod_{\substack{i=1 \\ i \neq j}}^N \frac{\sinh \gamma(u_j + u_i + 2) \sinh \gamma(u_j - u_i + 2)}{\sinh \gamma(u_j + u_i - 2) \sinh \gamma(u_j - u_i - 2)} \\ &= \prod_{k=1}^M \frac{\sinh \gamma(u_j - u_k^{(1)} + 1) \sinh \gamma(u_j + u_k^{(1)} + 1)}{\sinh \gamma(u_j - u_k^{(1)} - 1) \sinh \gamma(u_j + u_k^{(1)} - 1)} \\ &\prod_{\alpha=a,b} \frac{\sinh \gamma(u_k^{(1)} - c_\alpha + 1) \sinh \gamma(u_k^{(1)} + c_\alpha + 1)}{\sinh \gamma(u_k^{(1)} - c_\alpha - 1) \sinh \gamma(u_k^{(1)} + c_\alpha - 1)} \\ &\times \prod_{j=1}^N \frac{\sinh \gamma(u_k^{(1)} - u_j + 1) \sinh \gamma(u_k^{(1)} + u_j + 1)}{\sinh \gamma(u_k^{(1)} - u_j - 1) \sinh \gamma(u_k^{(1)} + u_j - 1)} \\ &= \prod_{\substack{i=1 \\ l \neq k}}^M \frac{\sinh \gamma(u_k^{(1)} - u_l^{(1)} + 2) \sinh \gamma(u_k^{(1)} + u_l^{(1)} + 2)}{\sinh \gamma(u_k^{(1)} - u_l^{(1)} - 2) \sinh \gamma(u_k^{(1)} + u_l^{(1)} - 2)} \end{aligned}$$

with  $\gamma$  and  $E$  as before.



## 5. Conclusion

In this paper, we studied the integrability of the two cases of one-dimensional  $q$ -deformed  $t$ - $J$  models with boundary Kondo impurities. The eigenvalues of the Hamiltonian in each case are derived from commuting boundary transfer matrices and the Bethe ansatz equations are obtained by using the algebraic Bethe ansatz method. Taking the limit  $q \rightarrow 1$  in the Bethe ansatz equations (4.19) and (4.25), we recover the Bethe ansatz equations for the two cases of the one-dimensional  $gl(2|1)$  and  $su(3)$   $t$ - $J$  models with the boundary Kondo impurities described when  $s_\alpha = \frac{1}{2}$  in [20]. Nevertheless, it would be interesting to extend the analysis of Bariev *et al* [33] to the present case, which will allow us to extract some exact results about the physical aspects of the models.

After completion of this work, we noticed a preprint from Fan *et al* [34], in which a boundary Kondo impurities with arbitrary spin is solved in the one-dimensional generalized supersymmetric  $t$ - $J$  model. However, they did not present the Hamiltonian explicitly and only treated the case corresponding to  $U_q(gl(2|1))$ .

## Acknowledgment

This work is supported by the Australian Research Council.

## Appendix. Derivation of the non- $c$ -number boundary $K$ matrices for the one-dimensional $q$ -deformed $t$ - $j$ models with boundary Kondo impurities

In this appendix, we sketch the procedure of solving the graded RE of (2.8). To describe the one-dimensional  $q$ -deformed supersymmetric  $U_q(gl(2|1))$   $t$ - $J$  model with boundary Kondo impurities, it is reasonable to assume that

$$K_-(u) = \begin{pmatrix} \bar{A}(u) & \bar{B}(u) & 0 \\ \bar{C}(u) & \bar{D}(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (\text{A.1})$$

Choosing  $\bar{A}(u) = F^{-1}(u)A(u)$ ,  $\bar{B}(u) = F^{-1}(u)B(u)$ ,  $\bar{C}(u) = F^{-1}(u)C(u)$ ,  $\bar{D}(u) = F^{-1}(u)D(u)$ , then

$$K_-(u) \propto \begin{pmatrix} A(u) & B(u) & 0 \\ C(u) & D(u) & 0 \\ 0 & 0 & F(u) \end{pmatrix}. \quad (\text{A.2})$$

For the  $R$  matrix (3.4), one may get, from the graded RE (2.8), 33 functional equations, of which 11 are identities. After some algebraic analysis, together with the  $U_q(su(2))$  symmetry, we may assume that

$$\begin{aligned} A(u) &= \alpha(u) + \beta(u)\sigma^z & B(u) &= \gamma(u)\sigma^- \\ C(u) &= \gamma(u)\sigma^+ & D(u) &= \tilde{\alpha}(u) - \tilde{\beta}(u)\sigma^z. \end{aligned} \quad (\text{A.3})$$

There are two equations automatically satisfied, leaving only 20 equations left to be solved:

$$\begin{aligned} A(u_1)B(u_2) + B(u_1)D(u_2) &= A(u_2)B(u_1) + B(u_2)D(u_1) \\ C(u_1)A(u_2) + D(u_1)C(u_2) &= C(u_2)A(u_1) + D(u_2)C(u_1) \\ (q^{u_-} - 1)(A(u_1)B(u_2) + B(u_1)D(u_2)) &= (q^{u_+} - 1)(B(u_1)F(u_2) - q^{u_-}B(u_2)F(u_1)) \\ (q^{u_-} - 1)(A(u_2)B(u_1) + B(u_2)D(u_1)) &= (q^{u_+} - 1)(B(u_1)F(u_2) - q^{u_-}B(u_2)F(u_1)) \\ (q^{u_-} - 1)(C(u_1)A(u_2) + D(u_1)C(u_2)) &= (q^{u_+} - 1)(C(u_1)F(u_2) - q^{u_-}C(u_2)F(u_1)) \end{aligned}$$

$$\begin{aligned}
& (q^{u_-} - 1)(C(u_2)A(u_1) + D(u_2)C(u_1)) = (q^{u_+} - 1)(C(u_1)F(u_2) - q^{u_-}C(u_2)F(u_1)) \\
& (q^{u_-} - 1)(A(u_1)A(u_2) + B(u_1)C(u_2) - q^{u_+}F(u_1)F(u_2)) \\
& \quad = (q^{u_+} - 1)(A(u_1)F(u_2) - q^{u_-}A(u_2)F(u_1)) \\
& (q^{u_-} - 1)(A(u_2)A(u_1) + B(u_2)C(u_1) - q^{u_+}F(u_1)F(u_2)) \\
& \quad = (q^{u_+} - 1)(A(u_1)F(u_2) - q^{u_-}A(u_2)F(u_1)) \\
& (q^{u_-} - 1)(C(u_1)B(u_2) + D(u_1)D(u_2) - q^{u_+}F(u_1)F(u_2)) \\
& \quad = (q^{u_+} - 1)(D(u_1)F(u_2) - q^{u_-}D(u_2)F(u_1)) \\
& (q^{u_-} - 1)(C(u_2)B(u_1) + D(u_2)D(u_1) - q^{u_+}F(u_1)F(u_2)) \\
& \quad = (q^{u_+} - 1)(D(u_1)F(u_2) - q^{u_-}D(u_2)F(u_1)) \\
& (q^{u_-} - 1)((q^{u_+2} - 1)B(u_1)D(u_2) - (1 - q^2)A(u_1)B(u_2)) \\
& \quad = (q^{u_+} - 1)((q^{u_-2} - 1)D(u_2)B(u_1) + q^{u_-}(1 - q^2)D(u_1)B(u_2)) \\
& (q^{u_-} - 1)((q^{u_+2} - 1)D(u_2)C(u_1) - (1 - q^2)C(u_2)A(u_1)) \\
& \quad = (q^{u_+} - 1)((q^{u_-2} - 1)C(u_1)D(u_2) + q^{u_-}(1 - q^2)C(u_2)D(u_1)) \\
& (q^{u_-} - 1)((q^{u_+2} - 1)C(u_1)A(u_2) - q^{u_+}(1 - q^2)D(u_1)C(u_2)) \\
& \quad = (q^{u_+} - 1)((q^{u_-2} - 1)A(u_2)C(u_1) + (1 - q^2)A(u_1)C(u_2)) \\
& (q^{u_-} - 1)((q^{u_+2} - 1)A(u_2)B(u_1) - q^{u_+}(1 - q^2)B(u_2)D(u_1)) \\
& \quad = (q^{u_+} - 1)((q^{u_-2} - 1)B(u_1)A(u_2) + (1 - q^2)B(u_2)A(u_1)) \\
& (q^{u_-} - 1)((1 - q^2)(A(u_1)A(u_2) - q^{u_+}D(u_2)D(u_1)) \\
& \quad - (q^{u_+2} - 1)(B(u_1)C(u_2) - C(u_2)B(u_1))) \\
& \quad = (q^{u_+} - 1)(1 - q^2)(D(u_2)A(u_1) - q^{u_-}D(u_1)A(u_2)) \\
& (q^{u_-} - 1)((1 - q^2)(A(u_1)A(u_2) - q^{u_+}D(u_2)D(u_1)) \\
& \quad - (q^{u_+2} - 1)(B(u_2)C(u_1) - C(u_1)B(u_2))) \\
& \quad = (q^{u_+} - 1)(1 - q^2)(D(u_2)A(u_1) - q^{u_-}D(u_1)A(u_2)) \\
& (q^{u_+2} - 1)((1 - q^2)B(u_1)D(u_2) + (q^{u_-2} - 1)B(u_2)D(u_1)) \\
& \quad = q^{u_+}(1 - q^2)((q^{u_-2} - 1)A(u_2)B(u_1) + (1 - q^2)A(u_1)B(u_2)) \\
& \quad \quad + q^2(q^{u_+} - 1)(q^{u_-} - 1)D(u_1)B(u_2) \\
& (q^{u_+2} - 1)(q^{u_-}(1 - q^2)C(u_1)A(u_2) + (q^{u_-2} - 1)C(u_2)A(u_1)) \\
& \quad = q^{u_+}(1 - q^2)((q^{u_-2} - 1)D(u_2)C(u_1) + q^{u_-}(1 - q^2)D(u_1)C(u_2)) \\
& \quad \quad + q^2(q^{u_+} - 1)(q^{u_-} - 1)A(u_1)C(u_2) \\
& (q^{u_+2} - 1)(q^{u_-}(1 - q^2)A(u_2)B(u_1) + (q^{u_-2} - 1)A(u_1)B(u_2)) \\
& \quad = q^{u_+}(1 - q^2)((q^{u_-2} - 1)B(u_1)D(u_2) + q^{u_-}(1 - q^2)B(u_2)D(u_1)) \\
& \quad \quad + q^2(q^{u_+} - 1)(q^{u_-} - 1)B(u_2)A(u_1) \\
& (q^{u_+2} - 1)((1 - q^2)D(u_2)C(u_1) + (q^{u_-2} - 1)D(u_1)C(u_2)) \\
& \quad = q^{u_+}(1 - q^2)((q^{u_-2} - 1)C(u_1)A(u_2) + (1 - q^2)C(u_2)A(u_1)) \\
& \quad \quad + q^2(q^{u_+} - 1)(q^{u_-} - 1)C(u_2)D(u_1)
\end{aligned}$$

with  $u_+ = u_1 + u_2$ ,  $u_- = u_1 - u_2$ . Substituting (A.3) into these equations, we find that all these equations are reduced to the following 11 equations:

$$\begin{aligned}
& (\alpha(u_1) - \beta(u_1))\gamma(u_2) + \gamma(u_1)(\tilde{\alpha}(u_2) - \tilde{\beta}(u_2)) \\
& \quad = (\alpha(u_2) - \beta(u_2))\gamma(u_1) + \gamma(u_2)(\tilde{\alpha}(u_1) - \tilde{\beta}(u_1)) \\
(1 - q^2)((1 - q^2)(\alpha(u_1) - \beta(u_1))\gamma(u_2) + (q^{u-+2} - 1)(\alpha(u_2) - \beta(u_2))\gamma(u_1)) \\
& \quad = (q^{u+2} - 1)((q^{u-+2} - 1)\gamma(u_2)(\tilde{\alpha}(u_1) - \tilde{\beta}(u_1)) \\
& \quad \quad + (1 - q^2)\gamma(u_1)(\tilde{\alpha}(u_2) - \tilde{\beta}(u_2))) \\
& \quad \quad - q^2(q^{u-} - 1)(q^{u+} - 1)(\tilde{\alpha}(u_1) + \tilde{\beta}(u_1))\gamma(u_2) \\
q^{u+}(1 - q^2)(q^{u-}(1 - q^2)\gamma(u_2)(\tilde{\alpha}(u_1) - \tilde{\beta}(u_1)) + (q^{u-+2} - 1)\gamma(u_1)(\tilde{\alpha}(u_2) - \tilde{\beta}(u_2))) \\
& \quad = (q^{u+2} - 1)((q^{u-+2} - 1)(\alpha(u_1) - \beta(u_1))\gamma(u_2) \\
& \quad \quad + q^{u-}(1 - q^2)(\alpha(u_2) - \beta(u_2))\gamma(u_1)) \\
& \quad \quad - q^2(q^{u-} - 1)(q^{u+} - 1)\gamma(u_2)(\alpha(u_1) + \beta(u_1)) \\
(q^{u+} - 1)((q^{u-+2} - 1)\gamma(u_1)(\alpha(u_2) + \beta(u_2)) + (1 - q^2)\gamma(u_2)(\alpha(u_1) + \beta(u_1))) \\
& \quad = (q^{u-} - 1)((q^{u+2} - 1)(\alpha(u_2) - \beta(u_2))\gamma(u_1) \\
& \quad \quad - q^{u+}(1 - q^2)\gamma(u_2)(\tilde{\alpha}(u_1) - \tilde{\beta}(u_1))) \\
(q^{u+} - 1)((q^{u-+2} - 1)(\tilde{\alpha}(u_2) + \tilde{\beta}(u_2))\gamma(u_1) + q^{u-}(1 - q^2)(\tilde{\alpha}(u_1) + \tilde{\beta}(u_1))\gamma(u_2)) \\
& \quad = (q^{u-} - 1)((q^{u+2} - 1)\gamma(u_1)(\tilde{\alpha}(u_2) - \tilde{\beta}(u_2)) \\
& \quad \quad - (1 - q^2)(\alpha(u_1) - \beta(u_1))\gamma(u_2)) \\
(q^{u+} - 1)(1 - q^2)((\tilde{\alpha}(u_2) - \tilde{\beta}(u_2))(\alpha(u_1) + \beta(u_1)) - q^{u-}(\tilde{\alpha}(u_1) - \tilde{\beta}(u_1))(\alpha(u_2) + \beta(u_2))) \\
& \quad = (q^{u-} - 1)(1 - q^2)((\alpha(u_1) + \beta(u_1))(\alpha(u_2) + \beta(u_2)) - q^{u+}(\tilde{\alpha}(u_2) \\
& \quad \quad - \tilde{\beta}(u_2))(\tilde{\alpha}(u_1) - \tilde{\beta}(u_1))) + (q^{u-} - 1)(q^{u+2} - 1)\gamma(u_1)\gamma(u_2) \\
(q^{u+} - 1)(1 - q^2)((\tilde{\alpha}(u_2) + \tilde{\beta}(u_2))(\alpha(u_1) - \beta(u_1)) - q^{u-}(\tilde{\alpha}(u_1) + \tilde{\beta}(u_1))(\alpha(u_2) - \beta(u_2))) \\
& \quad = (q^{u-} - 1)(1 - q^2)((\alpha(u_1) - \beta(u_1))(\alpha(u_2) - \beta(u_2)) - q^{u+}(\tilde{\alpha}(u_2) \\
& \quad \quad + \tilde{\beta}(u_2))(\tilde{\alpha}(u_1) + \tilde{\beta}(u_1))) - (q^{u-} - 1)(q^{u+2} - 1)\gamma(u_1)\gamma(u_2) \\
(q^{u-} - 1)((\alpha(u_1) - \beta(u_1))(\alpha(u_2) - \beta(u_2)) - q^{u+}F(u_1)F(u_2) + \gamma(u_1)\gamma(u_2)) \\
& \quad = (q^{u+} - 1)((\alpha(u_1) - \beta(u_1))F(u_2) - q^{u-}(\alpha(u_2) - \beta(u_2))F(u_1)) \\
(q^{u-} - 1)((\tilde{\alpha}(u_1) - \tilde{\beta}(u_1))(\tilde{\alpha}(u_2) - \tilde{\beta}(u_2)) - q^{u+}F(u_1)F(u_2) + \gamma(u_1)\gamma(u_2)) \\
& \quad = (q^{u+} - 1)((\tilde{\alpha}(u_1) - \tilde{\beta}(u_1))F(u_2) - q^{u-}(\tilde{\alpha}(u_2) - \tilde{\beta}(u_2))F(u_1)) \\
(q^{u-} - 1)((\alpha(u_1) + \beta(u_1))(\alpha(u_2) + \beta(u_2)) - q^{u+}F(u_1)F(u_2)) \\
& \quad = (q^{u+} - 1)((\alpha(u_1) + \beta(u_1))F(u_2) - q^{u-}(\alpha(u_2) + \beta(u_2))F(u_1)) \\
(q^{u-} - 1)((\tilde{\alpha}(u_1) + \tilde{\beta}(u_1))(\tilde{\alpha}(u_2) + \tilde{\beta}(u_2)) - q^{u+}F(u_1)F(u_2)) \\
& \quad = (q^{u+} - 1)((\tilde{\alpha}(u_1) + \tilde{\beta}(u_1))F(u_2) - q^{u-}(\tilde{\alpha}(u_2) + \tilde{\beta}(u_2))F(u_1)).
\end{aligned}$$

Solving these equations using some nontrivial tricks of variable separation, we have

$$\begin{aligned}
\alpha(u) + \beta(u) &= \tilde{\alpha}(u) + \tilde{\beta}(u) = \frac{(q^{u+2} - q^c)(q^{u+2} - q^{-c})}{(q^{u+c} - q^2)(q^{u-c} - q^2)} \\
\alpha(u) - \beta(u) &= \frac{q^{2u+4} - q^{2u+2} + q^{2u} - q^{u+c+2} - q^{u-c+2} + q^2}{(q^{u+c} - q^2)(q^{u-c} - q^2)} \\
\tilde{\alpha}(u) - \tilde{\beta}(u) &= \frac{q^{2u+2} - q^{u+c+2} - q^{u-c+2} + q^4 - q^2 + 1}{(q^{u+c} - q^2)(q^{u-c} - q^2)}
\end{aligned}$$

$$\begin{aligned}
\gamma(u) &= \frac{q(q^2 - 1)(q^{2u} - 1)}{(q^{u+c} - q^2)(q^{u-c} - q^2)} \\
F(u) &= \frac{(1 - \xi q^{-u})(q^{u+2} - q^c)(q^{u+2} - q^{-c})}{(1 - \xi q^u)(q^{u+c} - q^2)(q^{u-c} - q^2)} \\
\xi &= q^{c+2} \quad \text{or} \quad \xi = q^{-c+2}.
\end{aligned} \tag{A.4}$$

Choosing  $\xi = q^{c+2}$ , then substituting these results into (A.3) and (A.1), we may establish the non- $c$ -number boundary  $K$  matrix  $K_-(u)$  (3.5). Using the same method, we may find other non- $c$ -number boundary  $K$  matrices  $K_+(u)$  (3.6) by solving the dual graded RE (2.11).

A similar construction also works for the one-dimensional  $q$ -deformed  $U_q(su(3))$   $t$ - $J$  model with boundary Kondo impurities with the quantum  $R$  matrix (3.8).

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